

The density of the solution to the stochastic transport equation with fractional noise

Christian Olivera ^{1,*} Ciprian A. Tudor ^{2,3, †}

¹ Departamento de Matemática, Universidade Estadual de Campinas,
13.081-970-Campinas-SP-Brazil.
colivera@ime.unicamp.br

² Laboratoire Paul Painlevé, Université de Lille 1
F-59655 Villeneuve d'Ascq, France.

³Academy of Economical Studies, Bucharest, Romania
tudor@math.univ-lille1.fr

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Abstract

We consider the transport equation driven by the fractional Brownian motion. We study the existence and the uniqueness of the weak solution and, by using the tools of the Malliavin calculus, we prove the existence of the density of the solution and we give Gaussian estimates from above and from below for this density.

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1 Introduction

The purpose of this paper is to study the probability law of the real-valued solution of the following stochastic partial differential equations

$$\begin{cases} du(t, x) + b(t, x) \nabla u(t, x) dt + \nabla u(t, x) \circ dB_t^H + F(t, u) dt = 0, \\ u(0, x) = u_0(x), \end{cases} \quad (1)$$

where $B_t^H = (B_t^{H_1}, \dots, B_t^{H_d})$ is a fractional Brownian motion (fBm) in \mathbb{R}^d with Hurst parameter $H = (H_1, \dots, H_d) \in [\frac{1}{2}, 1]^d$ and the stochastic integration is understood in the symmetric (Stratonovich) sense. The equation (1) is usually called the stochastic transport equation and arises as a prototype model in a wide variety of phenomena. Although we introduced (1) in a general form, we mention that some results will be obtained in dimension one.

The stochastic transport equation with standard Brownian noise has been first studied in the celebrated works by Kunita [11], [12] and more recently it has been the object of study for many authors. We refer, among many others, to [4], [7], [8], [14], [15], [19].

Our aim is to analyze the stochastic partial equation (1) when the driving noise is the fractional Brownian motion, including the particular case of the Brownian motion. We will first give, by interpreting the stochastic integral in (1) as a symmetric integral via regularization in the Russo-Vallois sense [21], an existence and uniqueness result for the weak solution to (1) via the so-called method of characteristics and we express the solution as the initial value applied to the inverse flow generated by the equation of characteristics. This holds, when $H_i = \frac{1}{2}, i = 1, \dots, d$ for any dimension d and in dimension $d = 1$ if the Hurst parameter is bigger than one half. Using this representation of the solution to (1), we study the existence and the Gaussian estimates for its density via the analysis of the dynamic of the inverse flow. A classical tools to study the absolute continuity of the law of random variables with respect to the Lebesgue measure is the Malliavin calculus. We refer to the monographs [17] or [23] for various applications of the Malliavin calculus to the existence and smoothness of the density of random variables in general, and of solutions to stochastic equations in particular.

We will prove the Malliavin differentiability of the solution to (7) by analyzing the dynamic of the inverse flow generated by the characteristics (9). Using a result in [16] we obtain, in dimension $d = 1$ upper and lower Gaussian bounds for the density of the solution to the transport equation. We are also able to find the explicit form of the density in dimension $d \geq 2$ when the driving noise is the standard Brownian motion and the drift is divergence-free (i.e. the divergence of the drift vanishes).

We organized our paper as follows. In Section 2 we recall the existence and uniqueness results for the solution to the transport equation driven by the standard Brownian motion. In Section 3, we analyze the weak solution to the transport equation when the noise is the fBm, via the method of characteristics. In Section 4 we study the Malliavin differentiability of the solution to the equation of characteristics and this will be applied in Section 4 to obtain the existence and the Gaussian estimates for the solution to the transport equation. In Section 6 we obtain an explicit formula for the density when the noise is the Wiener process and the drift is divergence-free.

2 Stochastic transport equation driven by standard Brownian motion

Throughout the paper, we will fix a probability space (Ω, \mathcal{F}, P) and a d -dimensional Wiener process $(B_t)_{t \in [0, T]}$ on this probability space. We will denote by $(\mathcal{F}_t)_{t \in [0, T]}$ the filtration generated by B .

We will start by recalling some known facts on the solution to the transport equation driven by a standard Wiener process in \mathbb{R}^d .

The equation (1) is interpreted in the strong sense, as the following stochastic integral equation

$$u(t, x) = u_0(x) - \int_0^t b(s, x) \nabla u(s, x) \, ds - \sum_{i=1}^d \int_0^t \partial_{x_i} u(s, x) \circ dB_s^i - \int_0^t F(t, u) \, ds \quad (2)$$

for $t \in [0, T]$ and $x \in \mathbb{R}^d$.

The solution to (1) is related with the so-called equation of characteristics. That is, for $0 \leq s \leq t$ and $x \in \mathbb{R}^d$, consider the following stochastic differential equation in \mathbb{R}^d

$$X_{s,t}(x) = x + \int_s^t b(r, X_{s,r}(x)) \, dr + B_t - B_s. \quad (3)$$

and denote by $X_t(x) := X_{0,t}(x)$, $t \in [0, T]$, $x \in \mathbb{R}^d$.

For $m \in \mathbb{N}$ and $0 < \alpha < 1$, let us assume the following hypothesis on b :

$$b \in L^1((0, T); C_b^{m, \alpha}(\mathbb{R}^d)) \quad (4)$$

where $C^{m, \alpha}(\mathbb{R}^d)$ denotes the class of functions of class C^m on \mathbb{R}^d such that the last derivative is Hölder continuous of order α .

Let us recall the definition of the stochastic flow (see e.g. [10]).

Definition 1 *A stochastic flow is a family of maps $(\Phi_{s,t} : \mathbb{R}^d \rightarrow \mathbb{R}^d)_{0 \leq s \leq t \leq T}$ such that*

- $\lim_{t \rightarrow s_+} \Phi_{s,t}(x) = x$ for every $x \in \mathbb{R}^d$.
- $\Phi_{u,t} \circ \Phi_{s,u} = \Phi_{s,t}$ if $0 \leq s \leq u \leq t$.

Note that in [10] the some measurability is also required in the definition of the flow but, since we are working later in the paper with non-semimartingales, we will omit it.

It is well known that under conditions (4), $X_{s,t}(x)$ is a stochastic flow of C^m -diffeomorphism (see for example [12] and [10]). Moreover, the inverse $Y_{s,t}(x) := X_{s,t}^{-1}(x)$ satisfies the following backward stochastic differential equation

$$Y_{s,t}(x) = x - \int_s^t b(r, Y_{r,t}(x)) dr - (B_t - B_s). \quad (5)$$

for every $0 \leq s \leq t \leq T$, see [8] or [12] pp. 234.

In order to get the solution of (1) via the stochastic characteristic method we consider the following ordinary differential equation

$$Z_t(r) = r + \int_0^t F(s, Z_s(r)) ds. \quad (6)$$

We have the following representation of the solution to the transport equation in terms of the initial data and of the inverse flow (5). We refer to e.g. [10] or [5], Section 3 for the proof.

Lemma 1 *Assume (4) for $m \geq 3$ and let $u_0 \in C^{m,\delta}(\mathbb{R}^d)$, $F \in L^\infty((0, T); C_b^m(\mathbb{R}^d))$. Then the Cauchy problem (2) has a unique solution $u(t, \cdot)$ for $0 \leq t \leq T$ such that it is a C^m -semimartingale which can be represented as*

$$u(t, x) = Z_t(u_0(X_t^{-1}(x))), \quad t \in [0, T], x \in \mathbb{R}^d$$

where Z is the unique solution to (6) and $X_t^{-1} = X_{0,t}^{-1} = Y_{0,t}$ for every $t \in [0, T]$.

3 The weak solution of the transport equation driven by fractional Brownian motion

We discuss in this section the existence, uniqueness and the representation of the solution to the standard equation driven by a fractional Brownian motion with Hurst

parameter bigger than one half. We refer to the last section (the Appendix) for the basic properties of this process. We will restrict throughout this section to the case $d = 1$ and we will use the concept of weak solution. We explain at the end of this section (see Remark 1) why we need to assume these restrictions.

Consider the following one-dimensional Cauchy problem: given an initial-data u_0 , find $u(t, x; \omega) \in \mathbb{R}$, satisfying

$$\begin{cases} \partial_t u(t, x; \omega) + \left(\partial_x u(t, x; \omega) (b(t, x) + \frac{dB_t^H}{dt}(\omega)) \right) = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (7)$$

with $T > 0$, $((t, x) \in U_T, \omega \in \Omega)$, where $U_T = [0, T] \times \mathbb{R}$, and $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a given vector field. The noise B^H is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ and the stochastic integral in (7) will be understood in the symmetric sense via regularization [21] or [22]. The fBm B^H is related to the Brownian motion B via (31).

Let us first recall the notion of weak solution to (7).

Definition 2 *A stochastic process $u \in L^\infty(\Omega \times [0, T] \times \mathbb{R})$ is called a weak L^p -solution of the Cauchy problem (7), when for any $\varphi \in C_c^\infty(\mathbb{R})$, $\int_{\mathbb{R}} u(t, x) \varphi(x) dx$ is an adapted real value process which has a continuous modification, finite covariation, and for all $t \in [0, T]$, we have P -almost surely*

$$\begin{aligned} \int_{\mathbb{R}} u(t, x) \varphi(x) dx &= \int_{\mathbb{R}} u_0(x) \varphi(x) dx + \int_0^t \int_{\mathbb{R}} u(s, x) b(s, x) \partial_x \varphi(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s, x) b'(s, x) \varphi(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x \varphi(x) dx d^\circ B_s^H. \end{aligned} \quad (8)$$

where $b'(s, x)$ denotes the derivative of $b(s, x)$ with respect to the variable x .

At this point, we need to recall the definition of the symmetric integral $d^\circ B^H$ that appears in (8). assume $(X_t)_{t \geq 0}$ is a continuous process and $(Y_t)_{t \geq 0}$ is a process with paths in $L_{loc}^1(\mathbb{R}^+)$, i.e. for any $b > 0$, $\int_0^b |Y_t| dt < \infty$ a.s. The generalized stochastic integrals (forward, backward and symmetric) are defined through a regularization procedure see [21], [22]. That is, let $I^0(\epsilon, Y, dX)$ be the ϵ -symmetric integral

$$I^0(\epsilon, Y, dX) = \int_0^t Y_s \frac{(X_{s+\epsilon} - X_{s-\epsilon})}{2\epsilon} ds \quad t \geq 0.$$

The symmetric integral $\int_0^t Y d^\circ X$ is defined as

$$\int_0^t Y d^\circ X := \lim_{\epsilon \rightarrow 0} I^0(\epsilon, Y, dX)(t),$$

for every $t \in [0, T]$, provided the limit exist ucp (uniformly on compacts in probability).

Similarly to Lemma 1, we also have a representation formula for the weak solution in terms of the initial condition u_0 and the (inverse) stochastic flow associated to SDE (9).

Theorem 1 *Assume that $b \in L^\infty((0, T); C_b^1(\mathbb{R}^d))$. Then there exists a $C^1(\mathbb{R})$ stochastic flow of diffeomorphism $X_{s,t}, 0 \leq s \leq t \leq T$ that satisfies*

$$X_{s,t}(x) = x + \int_s^t b(u, X_{s,u}(x)) du + B_t^H - B_s^H \quad (9)$$

for every $x \in \mathbb{R}^d$. Moreover, if $d = 1$, given $u_0 \in L^\infty(\mathbb{R})$, the stochastic process

$$u(t, x) := u_0(X_t^{-1}(x)), \quad t \in [0, T], x \in \mathbb{R} \quad (10)$$

is the unique weak L^∞ -solution of the Cauchy problem (7), where $X_t := X_{0,t}$ for every $t \in [0, T]$.

Proof: We will proceed in several steps: first we show that (9) is a diffeomorphism flow, then we prove the uniqueness of the L^∞ weak solution to (7) and then we show that (10) satisfies the transport equation (7).

Let us first show that (9) generates a flow of diffeomorphism. By doing the linear transformation

$$Z_{s,t} = X_{s,t}(x) - (B_t^H - B_s^H)$$

we deduce that the equation (9) is equivalent to the random equation

$$Z_{s,t}(x) = x + \int_s^t b(r, Z_{s,r}(x) + B_r^H - B_s^H) dr \quad (11)$$

for $0 \leq s \leq t \leq T$.

From the classical theory for ordinary differential equations (see e.g. [2]) we have that $Z_{s,t}(x)$ with $0 \leq s \leq t \leq T$ is a $C^1(\mathbb{R}^d)$ diffeomorphism flow. Thus we deduce that $X_{s,t}(x)$ is a $C^1(\mathbb{R}^d)$ diffeomorphism flow.

In a second step, we will show that the transport equation with fBm noise admits a unique L^∞ weak solution. By linearity we have to show that a weak

L^∞ -solution with initial condition $u_0 = 0$ vanishes identically. Applying the Itô-Ventzel for the symmetric integral formula (see Proposition 9 of [9]) to $F(y) = \int u(t, x)\varphi(x - y) dx$ (which depends on ω), we obtain that

$$\begin{aligned} \int_{\mathbb{R}} u(t, x)\varphi(x - B_t^H)dx &= \int_0^t \int_{\mathbb{R}} b(s, x)\partial_x\varphi(x - B_s^H)u(s, x)dxds \\ &\quad + \int_0^t \int_{\mathbb{R}} b'(s, x)\varphi(x - B_s^H)u(s, x)dxds \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s, x)\partial_x\varphi(x - B_s^H)dx d^\circ B_s^H \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s, x)\partial_y[\varphi(x - B_s^H)]dx d^\circ B_s^H. \end{aligned} \quad (12)$$

We observe that $\partial_y[\varphi(x - B_s^H)] = -\partial_x\varphi(x - B_s^H)$. Thus the process

$$V(t, x) := u(t, x + B_t^H)$$

verifies

$$\begin{aligned} \int_{\mathbb{R}} V(t, x)\varphi(x)dx &= \int_0^t \int_{\mathbb{R}} b(s, x + B_s^H)\partial_x\varphi(x)V(s, x)dxds \\ &\quad + \int_0^t \int_{\mathbb{R}} b'(s, x + B_s^H)\varphi(x)V(s, x)dxds. \end{aligned}$$

Let ϕ_ε be a standard mollifier and let $V_\varepsilon(t, x) := V(t, \cdot) * \phi_\varepsilon$. Then it holds

$$\begin{aligned} \int_{\mathbb{R}} V(t, z)\phi_\varepsilon(x - z)dz &= \int_0^t \int_{\mathbb{R}} V(s, z)b(s, z + B_s^H)\partial_z\phi_\varepsilon(x - z)dzds \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s, z)b'(s, z + B_s^H)\phi_\varepsilon(x - z)dzds \end{aligned}$$

From an algebraic convenient manipulatio we get

$$\frac{dV_\varepsilon}{dt} - b(t, x - B_t^H)\partial_x V_\varepsilon = \mathcal{R}_\varepsilon(b, u)$$

where $\mathcal{R}_\varepsilon(b, u)$ is the commutator defined as

$$\mathcal{R}_\varepsilon(b, u) = (b\partial_x)(\phi_\varepsilon * u) - \phi_\varepsilon * ((b\partial_x)u).$$

Since $b(s, x + B_s^H)$ belongs a.s to $L^\infty((0, T); C_b^1(\mathbb{R}))$ then by the Commuting Lemma (see Lemma II.1 of [6]), the process $V_\varepsilon(t, x) = V(t, \cdot) * \phi_\varepsilon$ satisfies

$$\lim_{\varepsilon \rightarrow 0} \frac{dV_\varepsilon}{dt} - b(t, x - B_t^H) \partial_x V_\varepsilon = 0 \text{ a.s. in } L^1([0, T], L_{loc}^1(\mathbb{R})).$$

We deduce that if $\beta \in C^1(\mathbb{R})$ and β' is bounded, then

$$\frac{d\beta(V)}{dt} - b(t, x - B_t^H) \partial_x \beta(V) = 0. \quad (13)$$

Now, by Theorem II. 2 of [6], we define for each $M \in [0, \infty)$ the function $\beta_M(t) = (|t| \wedge M)^p$ and obtain that

$$\frac{d}{dt} \int \beta_M(V(t, x)) dx \leq C \int \beta_M(V(t, x)) dx.$$

Taking expectation we have that

$$\frac{d}{dt} \int \mathbb{E}(\beta_M(V(t, x))) dx \leq C \int \mathbb{E}(\beta_M(V(t, x))) dx.$$

From Gronwall Lemma we conclude that $\beta_M(V(t, x)) = 0$. Thus $u = 0$.

Let us finally show that (10) satisfies (7). We have that, by the change of variables $X_t^{-1}(y) = x$

$$\int_{\mathbb{R}} u_0(X_t^{-1})(y) \varphi(y) dy = \int_{\mathbb{R}^d} u_0(x) X_t'(x) \varphi(X_t(x)) dx, \quad (14)$$

for each $t \in [0, T]$, where $X_t'(x)$ denotes the derivative with respect to x of $X_t(x)$.

Notice that $X_t'(x) = 1 + \int_0^t b'(s, X_s(x)) X_s'(x) ds$ for every $t \in [0, T], x \in \mathbb{R}$. By applying Itô's formula (see [22], [21]) to the product

$$X_t'(x) \varphi(X_t(x))$$

and using the fact that B^H has zero quadratic variation when $H > \frac{1}{2}$ we obtain that

$$\begin{aligned} \int_{\mathbb{R}} u_0(X_t^{-1}(x)) \varphi(x) dx &= \int_{\mathbb{R}} u_0(x) dx + \int_0^t \int_{\mathbb{R}} u_0(x) b(s, X_s(x)) X_t'(x) \cdot \varphi'(X_s(x)) dx ds \\ &+ \int_0^t \int_{\mathbb{R}} u_0(x) b'(s, X_s(x)) X_t'(x) \varphi'(X_s(x)) dx ds \\ &+ \int_0^t \int_{\mathbb{R}} u_0(x) X_t'(x) \varphi'(X_s(x)) dy d^\circ B_s^H. \end{aligned} \quad (16)$$

Note that the Itô formula in [22] guarantees the existence of the symmetric stochastic integrals in (16) above. Now, by the change variable $y = X_t(x)$ we have that

$$\begin{aligned} \int_{\mathbb{R}} u_0(X_t^{-1}(x)) \varphi(x) dx &= \int_{\mathbb{R}} u_0(x) dx + \int_0^t \int_{\mathbb{R}} u_0(X_s^{-1}(x)) b(s, y) \cdot \varphi'(y) dy ds \\ &+ \int_0^t \int_{\mathbb{R}} u_0(X_s^{-1}(x)) b'(s, y) \varphi'(y) dy ds \\ &+ \int_0^t \int_{\mathbb{R}} u_0(X_s^{-1}(x)) \varphi'(y) dy d^\circ B_s^H. \end{aligned}$$

From this we conclude que $u(t, x) = u_0(X_t^{-1}(x))$ is a weak solution of (7). Its adaptedness is a consequence of (31). Thus the unique solution to (7) is $u(t, x) = u_0(X_t^{-1}(x))$ for every $t \in [0, T]$ and for every $x \in \mathbb{R}$. ■

Remark 1 • *We need to restrict to the situation $d = 1$ in order to get the existence of the symmetric integral in (16) or (12). Here we also used the hypothesis $H > \frac{1}{2}$ that ensures that there is not a second derivative term in the Itô formula.*

- *The uniqueness of the weak solution can be obtained with weaker assumption on the drift b by following the proof of Theorem 3.1 in [4].*

4 Fractional Brownian flow

In this section we will analyze the properties of the stochastic flow generated by the fractional Brownian motion. We will call it the fractional Brownian flow in the sequel. Fix $d \geq 1$ and let $B^H = (B^{H_1}, B^{H_2}, \dots, B^{H_d})$ be a d -dimensional fractional Brownian motion with Hurst parameter $H = (H_1, H_2, \dots, H_d) \in (0, 1)^d$.

Recall (see Theorem 1) that if $b \in L^\infty((0, T), C_b^1(\mathbb{R}^d))$, (9) generates a C^1 -stochastic flow of diffeomorphism. We next describe the dynamic of the inverse flow of (9).

Lemma 2 *Let $b \in L^\infty((0, T), C_b^1(\mathbb{R}^d))$ and denote, for every $0 \leq s \leq t \leq T$ and for every $x \in \mathbb{R}^d$*

$$Y_{s,t}(x) = X_{s,t}^{-1}(x) \tag{17}$$

the inverse of the stochastic flow given by (9). Then the inverse flow satisfies the backward stochastic equation

$$Y_{s,t}(x) = x - \int_s^t b(r, Y_{r,t}) dr - (B_t^H - B_s^H) \tag{18}$$

for every $x \in \mathbb{R}^d$.

Proof: It follows from Kunita [10]. Indeed, Lemms 6.2, page 235 in [10] says that for any continuous function in two variables g we have

$$\int_s^t g(r, X_{s,r}(y)) dr|_{y=X_{s,t}^{-1}(x)} = \int_s^t g(r, X_{r,t}^{-1}(x)) dr$$

and it suffices to apply the above identity to (9). ■

We need the following useful lemma.

Lemma 3 *Let us introduce the notation, for $t \in [0, T]$ and $x \in \mathbb{R}^d$,*

$$R_{t,x}(u) = Y_{t-u,t}(x), \quad \text{if } u \in [0, t]. \quad (19)$$

Then we have, for every $t \in [0, T]$, $u \in [0, t]$ and $x \in \mathbb{R}^d$

$$R_{t,x}(u) = x - \int_0^u b(a, R_{t,x}(a)) da - (B_t^H - B_{t-u}^H). \quad (20)$$

Proof: In (17) we use the change of notation $u = t - s$ and we get for every $y \in \mathbb{R}^d$,

$$R_{t,y}(u) = y - \int_{t-u}^t b(r, Y_{r,t}(x)) dr - (B_t^H - B_{t-u}^H)$$

and then, with the change of variables $a = t - r$ in the integral dr , we can write

$$R_{t,y}(u) = y - \int_0^u b(a, Y_{t,y}(a)) da - (B_t^H - B_{t-u}^H)$$

with $R_{u,y}(u) = y$. ■

As a consequence of the above Lemma 3, we get the uniqueness of solution to the backward equation (18) satisfied by the inverse flow.

Corollary 1 *If $(\tilde{Y}_{s,t})_{0 \leq s \leq t \leq T}$ is another two parameter process that satisfies (17) with $\tilde{Y}_{s,s}(x) = x$ and b is Lipschitz in x uniformy with respect to t , then $\tilde{Y}_{s,t}(x) = Y_{s,t}(x)$ for every $0 \leq s \leq t$ and for every $x \in \mathbb{R}^d$.*

Proof: If \tilde{Y} satisfies (17), then, if we denote $\tilde{R}_{t,x}(u) = \tilde{Y}_{t-u,t}(x)$, we get from Lemma that \tilde{R} satisfies (19) and the Gronwall lemma and the Lipschitz assumption on the drift b imply the conclusion. ■

We denote by D the Malliavin derivative with respect with the fBm B^H (see the Appendix).

Proposition 1 Assume $b \in L^\infty((0, T), C_b^1(\mathbb{R}^d))$ and let $X_{s,t}$ be given by (9). Then, for every $0 \leq s \leq t \leq T$ and for every $x \in \mathbb{R}^d$, the components of inverse flow $Y_{s,t}^i$ ($1 \leq i \leq d$) are Malliavin differentiable and for every $\alpha \in [s, t]$

$$D_\alpha Y_{s,t}^i(x) = - \int_s^t \sum_{j=1}^d \frac{\partial b^i}{\partial x_j}(r, Y_{r,t}) D_\alpha Y_{r,t}^j(x) dr - 1$$

and $D_\alpha Y_{s,t}^i(x) = 0$ if $\alpha \notin [s, t]$. We denoted by b^i ($1 \leq i \leq d$) the components of the vector mapping b .

Proof: It suffices to show that the random variable $R_{t,x}(u)$ defined by (19) is Malliavin differentiable for any $x \in \mathbb{R}^d$ and for every $0 \leq u \leq t \leq T$. We will give the sketsch of the proof which follows by a routine fix point argument. Fix $x \in \mathbb{R}^d, t \in [0, T]$ and define the iterations

$$R_{t,x}^{(0)}(u) = x, \text{ for every } u \in [0, t]$$

and for $n \geq 1$,

$$R_{t,x}^{(n)}(u) = x - \int_0^u b(a, R_{t,x}^{(n-1)}(a)) da - (B_t^H - B_{t-u}^H).$$

By induction, we can prove by standard arguments (see e.g. [17], Theorem 2.2.1) that for every $p \geq 1$

$$\sup_{0 \leq u \leq t} \mathbf{E} \left| R_{t,x}^{(n)}(u) \right|^p < \infty,$$

$$R_{t,x}^{(n),j}(u) \in \mathbb{D}^{1,\infty}, \quad j = 1, \dots, d,$$

and

$$\sup_{n \geq 1} \sup_{\alpha \in [0, T]} \mathbf{E} \left| D_\alpha R_{t,x}^{(n),j}(u) \right|^p < \infty$$

where $R_{t,x}^{(n),j}(u)$ denotes the j th component of $R_{t,x}^{(n)}(u)$. Moreover, the sequence of random variables $(R_{t,x}^{(n)}(u))_{n \geq 1}$ converges in L^p to $R_{t,x}(u)$ which is the unique solution to (20). It follows from Lemma 1.2.3 in [17] that $R_{t,x}(u)$ belongs to $\mathbb{D}^{1,\infty}$. ■

Remark 2 Note that, when the noise is the standard Brownian motion, the Malliavin differentiability of Y is also claimed in [15].

5 Existence and Gaussian bounds for the density of the solution to the transport equation in dimension one

In this section we will assume that $d = 1$. On the other hand, the results in these section (except Theorem 4) will hold for every $H \in (0, 1)$. We also mention that we will use the notation $c, C..$ for generic positive constants that may vary from line to line.

From Proposition 1 we immediately obtain the explicit expression for the Malliavin derivative of the inverse flow.

Proposition 2 *If $b \in L^\infty((0, T), C_b^1(\mathbb{R}^d))$ and $Y_{s,t}$ is defined by (17), we have for every α and for every $0 \leq s \leq t \leq T$*

$$D_\alpha Y_{s,t}(x) = -1_{[s,t]}(\alpha) e^{-\int_s^\alpha b'(r, Y_{r,t}) dr} \quad (21)$$

with $b'(t, x)$ the derivative of $b(t, x)$ with respect to x .

Proof: For every α , we have

$$D_\alpha Y_{s,t}(x) = - \int_s^t b'(s_1, Y_{s_1,t}(x)) D_\alpha Y_{s_1,t}(x) ds_1 - 1_{[s,t]}(\alpha)$$

and by iterating the above relation we can write, for every $0 \leq s \leq t \leq T$ and for every $\alpha \in [0, T]$,

$$\begin{aligned} D_\alpha Y_{s,t}(x) &= -1_{[s,t]}(\alpha) \sum_{n \geq 0} (-1)^n \int_s^\alpha ds_1 \int_{s_1}^\alpha ds_2 \dots \int_{s_{n-1}}^\alpha ds_n \\ &\quad \times b'(s_1, Y_{s_1,t}(x)) b'(s_2, Y_{s_2,t}(x)) \dots b'(s_n, Y_{s_n,t}(x)) \\ &= -1_{[s,t]}(\alpha) \sum_{n \geq 0} \frac{(-1)^n}{n!} \left(\int_s^t dr b'(r, Y_{r,t}(x)) \right)^n \\ &= -1_{[s,t]}(\alpha) e^{-\int_s^\alpha b'(r, Y_{r,t}) dr}. \end{aligned}$$

■

The main tool in order to obtain the Gaussian estimates for the density of the solution to the transport equation is the following result given in [16].

Proposition 3 *If $F \in \mathbb{F}^{1,2}$, let*

$$g_F(F) = \int_0^\infty d\theta e^{-\theta} \mathbf{E} \left[\mathbf{E}' \left(\langle DF, \widetilde{DF} \rangle_{\mathcal{H}} | F \right) \right]$$

where for any random variable X , we denoted

$$\tilde{X}(\omega, \omega') = X(e^{-\theta}w + \sqrt{1 - e^{-2\theta}}\omega').$$

Here \tilde{X} is defined on a product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}, P \times P')$ and \mathbf{E}' denotes the expectation with respect to the probability measure P' . If there exists two constants γ_{\min} and γ_{\max} such that almost surely

$$0 \leq \gamma_{\min} \leq g_F(F) \leq \gamma_{\max}$$

then F admits a density ρ . Moreover, for every $z \in \mathbb{R}$,

$$\frac{\mathbf{E}|F - \mathbf{E}F|}{2\gamma_{\max}^2} e^{-\frac{(z - \mathbf{E}F)^2}{2\gamma_{\min}^2}} \leq \rho(z) \leq \frac{\mathbf{E}|F - \mathbf{E}F|}{2\gamma_{\min}^2} e^{-\frac{(z - \mathbf{E}F)^2}{2\gamma_{\max}^2}}$$

To apply the above result, we need to controll the Malliavin derivative of the inverse flow. This will be done in the next result. Notice that a similar method has been used in e.g. [1], [3] or [18] for various types of stochastic equations. In the sequel \mathcal{H} denotes the canonical Hilbert space associated to the fractional Brownian motion (see the Appendix).

Proposition 4 Assume $H > \frac{1}{2}$ and $b \in L^\infty((0, T); C_b^1(\mathbb{R}))$. Then there exist two positive constants $c < C$ such that for every $t \in [0, T]$ and for every $x \in \mathbb{R}$

$$ct^{2H} \leq \langle DY_{0,t}(x), \widetilde{DY_{0,t}(x)} \rangle_{\mathcal{H}} \leq Ct^{2H} \quad (22)$$

where $Y_{0,t}$ is given by (18).

Proof: Assume $H = \frac{1}{2}$. Then $H = L^2([0, T])$ and

$$\langle DY_{0,t}(x), \widetilde{DY_{0,t}(x)} \rangle_{\mathcal{H}} = \int_0^t d\alpha e^{-\int_0^\alpha b'(r, Y_{r,t}(x))dr} e^{-\int_0^\alpha b'(r, \widetilde{Y_{r,t}(x)})dr}$$

and since

$$e^{-T\|b'\|_\infty} \leq e^{-\int_s^\alpha b'(r, Y_{r,t})dr} \leq e^{T\|b'\|_\infty} \quad (23)$$

(and a similar bound holds for the tilde process) we obtain

$$ct \leq \langle DY_{0,t}(x), \widetilde{DY_{0,t}(x)} \rangle_{\mathcal{H}} \leq Ct$$

with two positive constant c and C .

Assume $H > \frac{1}{2}$. Then by (30)

$$\langle DY_{0,t}(x), \widetilde{DY_{0,t}(x)} \rangle_{\mathcal{H}} = \alpha_H \int_0^t d\alpha \int_0^t d\beta e^{-\int_0^\alpha b'(r, Y_{r,t}(x)) dr} e^{-\int_0^\beta b'(r, \widetilde{Y_{0,t}(x)}) dr} |\alpha - \beta|^{2H-2}$$

and inequality (23) implies that

$$c \int_0^t d\alpha \int_0^t d\beta |\alpha - \beta|^{2H-2} \leq \langle DY_{0,t}(x), \widetilde{DY_{0,t}(x)} \rangle_{\mathcal{H}} \leq C \int_0^t d\alpha \int_0^t d\beta |\alpha - \beta|^{2H-2}$$

which immediately gives (22).

Assume $H < \frac{1}{2}$. Then Proposition 23 in [3] implies the lower bound in (22). Concerning the upper bound, it suffices again to follow [3], Section 3.4 and to note that for every $\alpha, \beta \in [0, T]$ with $\alpha > \beta$ we have

$$|D_\alpha Y_{0,t}(x) - D_\beta Y_{0,t}(x)| \leq e^{-\int_0^\beta b'(r, Y_{r,t}) dr} \left| e^{-\int_\beta^\alpha b'(r, Y_{r,t}) dr} - 1 \right| \leq c(\alpha - \beta)$$

and the same bound holds for the $\widetilde{Y_{0,t}}$. ■

Denote by $m := \mathbf{E}u(t, x)$ (it satisfies a parabolic equation, see e.g. [8]). We are ready to state our main result.

Theorem 2 *Let $u(t, x)$ be the solution to the transport equation (7). Assume that $u_0 \in C^1(\mathbb{R})$ such that there exist $0 < c < C$ with $c \leq u'_0(x) < C$ for every $x \in \mathbb{R}$ and $b \in L^\infty((0, T); C_b^1(\mathbb{R}))$. Then, for every $t \in [0, T]$ and for every $x \in \mathbb{R}$, the random variable $u(t, x)$ is Malliavin differentiable. Moreover $u(t, x)$ admits a density $\rho_{u(t, x)}$ and there exist two positive constants c_1, c_2 such that*

$$\frac{\mathbf{E}|u(t, x) - m|}{2c_1 t^{2H}} e^{-\frac{(y-m)^2}{2c_2 t^{2H}}} \leq \rho_{u(t, x)} \leq \frac{\mathbf{E}|u(t, x) - m|}{2c_2 t^{2H}} e^{-\frac{(y-m)^2}{2c_1 t^{2H}}} \quad (24)$$

Proof: Since by Theorem 1, $u(t, x) = u_0(Y_{0,t}(x))$, we get the Malliavin differentiability of $u(t, x)$ from Proposition 1 and the chain rule for the Malliavin derivative (see e.g. [17]). Moreover, the chain rule implies

$$D_\alpha u(t, x) = u'_0(Y_{0,t}(x)) D_\alpha Y_{0,t}(x)$$

and thus

$$\langle Du(t, x), \widetilde{Du(t, x)} \rangle_{\mathcal{H}} = u'_0(Y_{0,t}(x)) u'_0(\widetilde{Y_{0,t}(x)}) \langle DY_{0,t}(x), \widetilde{DY_{0,t}(x)} \rangle_{\mathcal{H}}.$$

By Proposition 4 and the assumption $u_0 \in C_b^1$, there exists two strictly positive constants $c < C$ such that

$$ct^{2H} \leq \langle Du(t, x), \widetilde{Du(t, x)} \rangle_{\mathcal{H}} \leq Ct^{2H}$$

for every $t \in [0, T]$ and for every $x \in \mathbb{R}$. Now, Proposition 3, point 2. implies that, if $F = u(t, x)$ then

$$c_1 t^{2H} \leq g_F(F) \leq c_2 t^{2H}$$

and Proposition 3, point 1. gives the conclusion. ■

6 Explicit expression of the density when the noise is the Brownian motion in \mathbb{R}^d

We obtained above the existence and Gaussian estimate for the solution to the transport equation in dimension 1 and for $H \geq \frac{1}{2}$. In this section, we will assume $d \geq 2$, $H = \frac{1}{2}$, that is, the transport equation is driven a standard Brownian motion in \mathbb{R}^d . We obtain the followin explicit expression for the density of the solution when the divergence of the drift b vanishes.

Theorem 3 *Assume $d \geq 2$ and let u_0 be a $C^{m, \delta}(\mathbb{R}^d)$ diffeomorphism. Assume (4) for $m \geq 3$. Moreover, suppose that*

$$\operatorname{div} b = 0. \tag{25}$$

Fix $t \in [0, T]$ and $x \in \mathbb{R}^d$. Then the law of the solution of (1), has a density $\tilde{\rho}$ with respect to the Lebesgue measure. Moreover the density $\tilde{\rho}$ admits the representation

$$\tilde{\rho} = Ju_0(Z_t^{-1}(y))JZ_t\rho(u_0^{-1}(Z_t^{-1}(y)), t, x) \tag{26}$$

where ρ denotes the density of the solution to (3).

Proof: Let $u(t, x)$ solution of the SPDE (1). By Lemma 1 we have that $u(t, x)$ has the representation

$$u(t, x) = Z_t(u_0(X_t^{-1}(x))).$$

Let ϕ_ε be a standard mollifier and consider a smooth function $\varphi \in C_c^\infty(\mathbb{R}^d)$. Then

$$\begin{aligned}
\mathbf{E}[\varphi(u(t, x))] &= \mathbf{E}[\varphi(Z_t(u_0(X_t^{-1}(x))))] \\
&= \mathbf{E} \left[\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^d} \phi_\epsilon(y - x) \varphi(Z_t(u_0(X_t^{-1}(y)))) dy \right] \\
&= \lim_{\epsilon \rightarrow 0} \mathbf{E} \left[\int_{\mathbb{R}^d} \phi_\epsilon(y - x) \varphi(Z_t(u_0(X_t^{-1}(y)))) dy \right].
\end{aligned}$$

The assumption (25) implies that $JX_t = 1$, where JX_t denote of the Jacobian map of X_t . By doing one more time a change of variable, we can write

$$\begin{aligned}
\mathbf{E}[\varphi(u(t, x))] &= \lim_{\epsilon \rightarrow 0} \mathbf{E} \left[\int_{\mathbb{R}^d} \phi_\epsilon(y - x) \varphi(Z_t(u_0(X_t^{-1}(y)))) dy \right] \\
&= \lim_{\epsilon \rightarrow 0} \mathbf{E} \left[\int_{\mathbb{R}^d} \phi_\epsilon(X_t(y) - x) \varphi(Z_t(u_0(y))) dy \right] \\
&= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^d} \mathbf{E}[\phi_\epsilon(X_t(y) - x)] \varphi(Z_t(u_0(y))) dy \tag{27}
\end{aligned}$$

The random variable $X_t(x)$ admits a density ρ in any dimension d . This is an easy consequence of equation (9) (see e.g. [17]). Therefore, (27) becomes

$$\begin{aligned}
\mathbf{E}[\varphi(u(t, x))] &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^d} \mathbf{E}[\phi_\epsilon(X_t(y) - x)] \varphi(Z_t(u_0(y))) dy \\
&= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_\epsilon(u - x) \rho(u, t, y) du \varphi(Z_t(u_0(y))) dy
\end{aligned}$$

and by calculating the limit above when $\epsilon \rightarrow 0$ we get

$$\mathbf{E}[\varphi(u(t, x))] = \int_{\mathbb{R}^d} \rho(y, t, x) \varphi(Z_t(u_0(y))) dy.$$

Finally, the making succesively the changes of variables $w = u_0(y)$ and $y = Z_t(w)$ we obtain

$$\begin{aligned}
\mathbf{E}[\varphi(u(t, x))] &= \int_{\mathbb{R}^d} Ju_0 \rho(u_0^{-1}(w), t, x) \varphi(Z_t(w)) dw \\
&= \int_{\mathbb{R}^d} Ju_0(Z_t^{-1}(y)) JZ_t \rho(u_0^{-1}(Z_t^{-1}(y)), t, x) \varphi(y) dy
\end{aligned}$$

and thus relation (26) is obtained. ■

Remark 3 *The assumption $\text{div} b = 0$ can be interpreted as follows (see [13]): in fluid mechanics or more generally in continuum mechanics, incompressible flow (isochoric flow) refers to a flow in which the material density is constant within a fluid parcelan infinitesimal volume that moves with the velocity of the fluid. This is equivalent to the condition that the divergence of the fluid velocity is zero.*

7 Appendix

We present here some basic element on the fractional Brownian motion and on the Malliavin calculus.

7.1 Fractional Brownian motion

Consider $(B_t^H)_{t \in [0, T]}$ a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Recall that it is a centered Gaussian process with covariance function

$$\mathbf{E} B_t^H B_s^H := R_H(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad s, t \in [0, T]. \quad (28)$$

The fractional Brownian motion can be also defined as the only self-similar Gaussian process with stationary increments.

Denote by \mathcal{H} its canonical Hilbert space. If $H = \frac{1}{2}$ then $B^{\frac{1}{2}}$ is the standard Brownian motion (Wiener process) W and in this case $\mathcal{H} = L^2([0, T])$. Otherwise \mathcal{H} is the Hilbert space on $[0, T]$ extending the set of indicator function $\mathbf{1}_{[0, t]}, t \in [0, T]$ (by linearity and closure under the inner product) the rule

$$\langle \mathbf{1}_{[0, s]}, \mathbf{1}_{[0, t]} \rangle_{\mathcal{H}} = R_H(s, t) := 2^{-1} \left(s^{2H} + t^{2H} - |t - s|^{2H} \right).$$

The followings facts will be needed in the sequel (we refer to [20] or [17] for their proofs):

- If $H > \frac{1}{2}$, the elements of \mathcal{H} may be not functions but distributions; it is therefore more practical to work with subspaces of \mathcal{H} that are sets of functions. Such a subspace is

$$|\mathcal{H}| = \left\{ f : [0, T] \rightarrow \mathbb{R} \mid \int_0^T \int_0^T |f(u)| |f(v)| |u - v|^{2H-2} dv du < \infty \right\}.$$

Then $|\mathcal{H}|$ is a strict subspace of \mathcal{H} and we actually have the inclusions

$$L^2([0, T]) \subset L^{\frac{1}{H}}([0, T]) \subset |\mathcal{H}| \subset \mathcal{H}. \quad (29)$$

- The space $|\mathcal{H}|$ is not complete with respect to the norm $\|\cdot\|_{\mathcal{H}}$ but it is a Banach space with respect to the norm

$$\|f\|_{|\mathcal{H}|}^2 = \int_0^T \int_0^T |f(u)| |f(v)| |u - v|^{2H-2} dv du.$$

- If $H > \frac{1}{2}$ and f, g are two elements in the space $|\mathcal{H}|$, their scalar product in \mathcal{H} can be expressed by

$$\langle f, g \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T du dv |u - v|^{2H-2} f(u) g(v) \quad (30)$$

where $\alpha_H = H(2H - 1)$.

- when $H < \frac{1}{2}$ then the canonical Hilbert space is a space of functions. We have

$$C^\gamma \subset \mathcal{H} \subset L^2([0, T])$$

for all $\gamma > \frac{1}{2} - H$ where C^γ denotes the class of Hölder continuous functions of order γ .

- The fBm admits a representation as Wiener integral of the form

$$B_t^H = \int_0^t K_H(t, s) dW_s, \quad (31)$$

where $W = \{W_t, t \in T\}$ is a Wiener process, and $K_H(t, s)$ is the kernel

$$K_H(t, s) = d_H (t - s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F_1\left(\frac{t}{s}\right), \quad (32)$$

d_H being a constant and

$$F_1(z) = d_H \left(\frac{1}{2} - H\right) \int_0^{z^{-1}} \theta^{H-\frac{3}{2}} \left(1 - (\theta + 1)^{H-\frac{1}{2}}\right) d\theta.$$

If $H > \frac{1}{2}$, the kernel K_H has the simpler expression

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \quad (33)$$

where $t > s$ and $c_H = \left(\frac{H(H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$.

A d dimensional fractional Brownian motion $B^H = (B^{H_1}, \dots, B^{H_d})$ with Hurst parameter $H = (H_1, \dots, H_d) \in (0, 1)^d$ is a centered Gaussian process in \mathbb{R}^d with independent components and the covariance of the i th component is given by

$$R_{H_i}(t, s) = \mathbf{E} B_t^{H_i} B_s^{H_i} = \frac{1}{2} (t^{2H_i} + s^{2H_i} - |t - s|^{2H_i})$$

for every $1 \leq i \leq d$.

7.2 The Malliavin derivative

Here we describe the elements from the Malliavin calculus that we need in the paper. We refer [17] for a more complete exposition. Consider \mathcal{H} a real separable Hilbert space and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space (Ω, \mathcal{A}, P) , which is a centered Gaussian family of random variables such that $\mathbf{E}(B(\varphi)B(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}$.

We denote by D the Malliavin derivative operator that acts on smooth functions of the form $F = g(B(\varphi_1), \dots, B(\varphi_n))$ (g is a smooth function with compact support and $\varphi_i \in \mathcal{H}, i = 1, \dots, n$)

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(B(\varphi_1), \dots, B(\varphi_n)) \varphi_i.$$

It can be checked that the operator D is closable from \mathcal{S} (the space of smooth functionals as above) into $L^2(\Omega; \mathcal{H})$ and it can be extended to the space $\mathbb{D}^{1,p}$ which is the closure of \mathcal{S} with respect to the norm

$$\|F\|_{1,p}^p = \mathbf{E}F^p + \mathbf{E}\|DF\|_{\mathcal{H}}^p.$$

We denote by $\mathbb{D}^{k,\infty} := \cap_{p \geq 1} \mathbb{D}^{k,p}$ for every $k \geq 1$. In our paper, \mathcal{H} will be the canonical Hilbert space associated with the fractional Brownian motion, as defined in the previous paragraph.

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